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The Waterbed Effect in Spectral Estimation

Let $\Phi(\omega)$ denote the power spectral density of a discrete-time stationary Gaussian signal, where $\omega \in [-\pi, \pi]$ is the normalized angular frequency variable. Let $\hat{\Phi}(\omega)$ denote a windowed periodogram estimate of $\Phi(\omega)$. Under mild conditions on $\Phi(\omega)$, the large-sample relative variance of $\hat{\Phi}(\omega)$ is given by

$$\text{relative variance}\left[\hat{\Phi}(\omega)\right] \triangleq \frac{\text{var}[\hat{\Phi}(\omega)]}{\Phi^2(\omega)} = \frac{M}{N} \quad (\text{for } N \gg 1) \tag{1}$$

where N is the number of data samples and M denotes the effective time length of the window used (with $M = N$ for the unwindowed periodogram and usually $M \ll N$ for the windowed versions) (see, e.g., [1]–[3]). Under the simplifying assumption that $\Phi(\omega)$ is a piecewise constant function over M subintervals of the frequency interval $[-\pi, \pi]$, it was shown in [4] that (1) can in fact be viewed as a type of Cramér-Rao bound (CRB) on the relative variance of a much larger class of nonparametric spectral estimation methods than just the windowed periodograms. In particular, this discussion implies that the (large-sample) average relative variance of a nonparametric spectral estimation method may be lower bounded by an expression of the form:

$$\text{average relative variance}\left[\hat{\Phi}(\omega)\right] \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\text{var}\left[\hat{\Phi}(\omega)\right]}{\Phi^2(\omega)} d\omega = \frac{M}{N}. \tag{2}$$

Next, consider the parametric spectral estimation of $\Phi(\omega)$. Let $\Phi(\omega, \boldsymbol{\theta})$ denote a known parameterization of $\Phi(\omega)$, where $\boldsymbol{\theta}$ is the vector of unknown parameters of dimension m,

$$\dim(\boldsymbol{\theta}) = m. \tag{3}$$

Also, let $\Phi(\omega, \hat{\boldsymbol{\theta}})$ denote the parametric estimate of $\Phi(\omega)$ corresponding to the estimate $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$. Under the assumption that $\Phi(\omega)$ is a rational function of ω and that

$$m \gg 1 \quad (\text{but } m/N \ll 1) \tag{4}$$

it follows from results in [5] (for the all-pole signal case) and [6] (for mixed pole-zero signals) that large-sample CRBs, similar to (1) and (2), hold true:

$$\frac{\text{var}[\Phi(\omega, \hat{\boldsymbol{\theta}})]}{\Phi^2(\omega)} = \frac{2m}{N} \quad (\text{for } m \gg 1, N \gg 1, m/N \ll 1) \tag{5}$$

and, as a corollary,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\text{var}[\Phi(\omega, \hat{\boldsymbol{\theta}})]}{\Phi^2(\omega)} d\omega = \frac{2m}{N} \quad (\text{for } m \gg 1, N \gg 1, m/N \ll 1). \tag{6}$$

However, the assumption that $m \gg 1$ made above is often impractical. Indeed, in many applications of parametric spectral estimation m takes on fairly small values (such as $m \leq 10$). This drawback of the analysis in [5] and [6] was noted in the recent paper [7] whose main goal was to verify whether the CRBs in (5) and (6) hold also for small m values. As shown in [7], (5) may be a poor approximation for small values of m. However, interestingly enough, (6) was shown to hold true for any value of m:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\text{var}[\Phi(\omega, \hat{\boldsymbol{\theta}})]}{\Phi^2(\omega)} d\omega = \frac{2m}{N} \quad \text{for } m \geq 1 \quad (\text{and } N \gg 1). \tag{7}$$

In particular, the above discussion implies that, while the curve of the CRB on the relative variance of $\Phi(\omega, \hat{\boldsymbol{\theta}})$ may change with the parameter values in $\boldsymbol{\theta}$, for small values of m, the area of this curve remains constant regardless of its shape. This behavior has been suggestively called a waterbed effect in [7].

In this lecture note, we present a textbook-like derivation of the waterbed effect result in (7). Compared with [7], our analysis is much simpler and yet slightly more general in that it is not limited to rational

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spectral densities as in [7] (note, however, that [7] has also derived a closed-form expression for the finite-m CRB on the relative variance of $\Phi(\omega, \hat{\boldsymbol{\theta}})$, which we do not).

Derivation of the Waterbed Effect Result

Under fairly general conditions on $\Phi(\omega)$ and $\hat{\boldsymbol{\theta}}$, we can use a Taylor series expansion of $\Phi(\omega, \hat{\boldsymbol{\theta}})$ around the true parameter vector to obtain

$$\Phi(\omega, \hat{\boldsymbol{\theta}}) \approx \Phi(\omega, \boldsymbol{\theta}) + \frac{\partial \Phi(\omega, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}^T}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \quad (\text{for } N \gg 1) \tag{8}$$

where $(\cdot)^T$ denotes the transpose, and $(\partial \Phi(\omega, \boldsymbol{\theta}))/(\partial \boldsymbol{\theta}^T)$ denotes the $1 \times m$ gradient vector. Hence, asymptotically in N, the CRB on the variance of $\Phi(\omega, \hat{\boldsymbol{\theta}})$ is given by

$$\mathrm{var}[\Phi(\omega, \hat{\boldsymbol{\theta}})] = \frac{\partial \Phi(\omega, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}^T} \times \mathbf{C} \frac{\partial \Phi(\omega, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \tag{9}$$

where $\mathbf{C}$ denotes the CRB on the covariance matrix of the parameter estimate vector, $\hat{\boldsymbol{\theta}}$. Under the Gaussianity assumption made above and some regularity conditions on $\Phi(\omega)$, the matrix $\mathbf{C}$ is given by the so-called Whittle's formula (see, e.g., [8], [3]):

$$\mathbf{C} = \left\{ \frac{N}{4\pi} \int_{-\pi}^{\pi} \frac{1}{\Phi^2(\omega)} \times \frac{\partial \Phi(\omega, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \frac{\partial \Phi(\omega, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}^T} d\omega \right\}^{-1}. \tag{10}$$

Combining (9) (rewritten as

$$\mathrm{var}[\Phi(\omega, \hat{\boldsymbol{\theta}})] = \mathrm{tr}\left[\mathbf{C} \frac{\partial \Phi(\omega, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \times \frac{\partial \Phi(\omega, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}^T} \right],$$

where $\mathrm{tr}(\cdot)$ denotes the trace operator) and (10) yields the following expression for the average relative variance:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{var}[\Phi(\omega, \hat{\boldsymbol{\theta}})]}{\Phi^2(\omega)} d\omega = \mathrm{tr}\left\{ \mathbf{C} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\Phi^2(\omega)} \times \frac{\partial \Phi(\omega, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \frac{\partial \Phi(\omega, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}^T} d\omega \right\} = \frac{2}{N} \mathrm{tr}\left(\mathbf{C}\mathbf{C}^{-1}\right) = \frac{2m}{N} \tag{11}$$

which proves the waterbed effect result in (7).

An Example

To emphasize the nature of this waterbed effect, we present a simple example. In the interest of clarity, consider the restricted case of a first-order real-valued AR model. Then drawing on the results in [7], we get

$$\frac{\mathrm{var}[\Phi(\omega, \hat{\boldsymbol{\theta}})]}{\Phi^2(\omega)} = \frac{2}{N} \left[\frac{1 - |\xi_1|^2}{|e^{j\omega} - \xi_1|^2} + \mathrm{Re}\left\{ \frac{1 - \xi_1^2}{(e^{j\omega} - \xi_1)^2} \right\} \right] \tag{12}$$

where ξ_1 is the true real-valued autoregrgrssive model pole. Note that this expression is different from (5), since the latter is an approximation that assumes m is large, while (12) does not.

In this case, the right-hand side of (12) makes explicit the nature of the waterbed effect. In particular, the denominator terms in (12) indicate that a pole near the unit circle will introduce peaks in the relative variance at frequencies near that pole. By the waterbed effect, these peaks will need to be balanced by smaller relative variance in other frequency regions. For instance, for the two cases of $\xi_1 = 0.95$ and $\xi_1 = 0.2$, the relative variances are shown in Figure 1. Note that, as just discussed, a pole at $\xi_1 = 0.95$ near the unit circle produces a large peak in relative variance, especially in comparison to the case $\xi_1 = 0.2$ where the pole is away from the unit circle. Furthermore, by the waterbed effect, the large peak in variance in the $\xi_1 = 0.95$ case is balanced by a smaller variance at other frequencies, and again especially in comparison to the case $\xi_1 = 0.2$.

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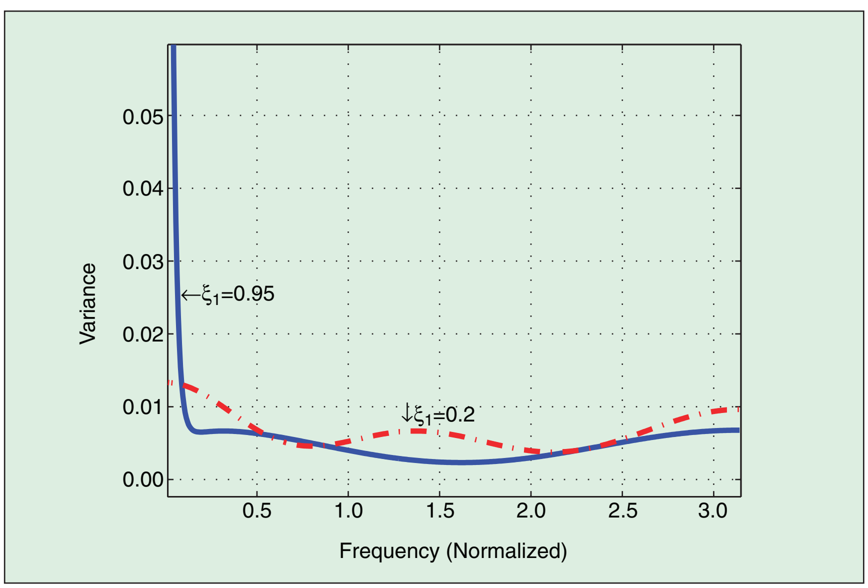

▲ *1. Relative variance of first-order real-valued AR spectral estimates for two cases of pole position ξ_1. Note the waterbed effect, in which increased variance at one frequency is balanced by decreased variance at others. Here the case of data length $N = 1,000$ is shown.*

Charles: I'm now working half time (or rather, half effort). I don't know when I will fully retire. My wife has been renovating a property on Cape Cod, which is nearly finished. I like to joke that it's rising, Phoenixlike, from the ashes of our savings. When I can spend some more time there, I want to try writing about how public policy decisions are made when they are based on technology.

SPM: *It has been an honor. Thank you and hope we will do it again.*

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If the reader has any comments regarding this article, please e-mail one of the authors. Feedback from our readers, either positive or negative, is most welcome.

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Concluding Remarks

In view of (2) and (7), the waterbed effect result, also called an uncertainty conservation result in [7], appears to be a fundamental property of both nonparametric and parametric spectral estimation methods. Consequently, an even "more intuitive" or "higher-level" derivation of this property than the one presented herein might exist, but it remains to be discovered.

Acknowledgments

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