Orthonormal Basis Functions for Continuous-Time Systems: Completeness and L_p–Convergence

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Abstract

In this paper, model sets for continuous–time linear time invariant systems that are spanned by fixed pole orthonormal bases are investigated. These bases generalise the well known Laguerre and Kautz bases. It is shown that the obtained model sets are complete in all of the Hardy spaces $H_p(\Pi)$, $1 \leq p < \infty$ and the right half plane algebra $A(\Pi)$ provided that a mild condition on the choice of basis poles is satisfied. As a further extension, the paper shows how orthonormal model sets, that are norm dense in $H_p(\Pi)$, $1 \leq p < \infty$ and which have a prescribed asymptotic order may be constructed. Finally, it is established that the Fourier series formed by orthonormal basis functions converge in all spaces $H_p(\Pi)$, $1 < p < \infty$. The results in this paper have application in system identification, model reduction and control system synthesis.

Keywords: Rational basis functions, orthonormal, continuous-time systems, Fourier series, L_p convergence.

1 Notation

- $\mathbf{C}$ the field of complex numbers.
- $\mathbf{R}$ the field of real numbers.
- Π the open right half plane $\{s \in \mathbf{C} : \mathrm{Re}\{s\} > 0\}$.
- $\overline{\Pi}$ the closed right half plane $\{s \in \mathbf{C} : \mathrm{Re}\{s\} \geq 0\}$.
- $\mathbf{D}$ the open unit disk $\{z \in \mathbf{C} : |z| < 1\}$.
- $\mathbf{T}$ the unit circle $\{z \in \mathbf{C} : |z| = 1\}$.
- $H_p(\Pi)$ the Hardy spaces of functions $f(s)$ analytic on Π and such that $\|f\|_p^p = (1/2\pi)\sup_{x>0}\int_{-\infty}^{\infty}|f(x+jy)|^p\,\mathrm{d}y < \infty$, $0 < p < \infty$ and $\|f\|_\infty = \sup_{s\in\Pi}|f(s)| < \infty$.
- $A(\Pi)$ the right half plane algebra $\{f : f \in H_\infty(\Pi)$ and continuous on $\overline{\Pi}\}$.
- $A(\mathbf{D})$ the disk algebra $\{f : f$ analytic on $\mathbf{D}$ and continuous on $\overline{\mathbf{D}}\}$.
- $\mathrm{sp}A$ the linear span of A.
- $\overline{a}$ the complex conjugate of a.
- $O(|s|^{-m})$ The notation $f(s) = O(|s|^{-m})$ as $|s| \to \infty$ means that

$$\limsup_{|s|\to\infty} |s|^m |f(s)| < \infty$$

2 Introduction

A fundamental idea in various areas of applied mathematics, control theory, signal processing and system analysis is that of decomposing (perhaps infinite dimensional) descriptions of linear time invariant dynamics in terms of an orthonormal basis. This approach is of greatest utility when accurate system descriptions are achieved with only a small number of basis functions. In recognition of this, there has been much work over the past several decades [18, 6, 10, 26, 27] and, with renewed interest, more recently [31, 30, 29, 13, 9, 20, 5, 19] on the construction, analysis and application of rational orthonormal bases suitable for providing linear system characterisations.

In a system theoretic context, the applications of these orthonormal basis ideas have been manifold, but nevertheless have concentrated mainly on the discrete time setting [29, 30, 25, 21, 22, 9, 19, 5]. Motivated largely by problems of estimation from frequency domain data [1, 17, 24, 8], but also with control system analysis and synthesis in mind [14, 12] this paper focuses attention on the continuous time scenario by considering the set of basis functions defined by a choice of numbers $\{a_k\} \in \Pi$ as

$$B_n(s) \triangleq \frac{\sqrt{2\mathrm{Re}\{a_n\}}}{s+a_n}\varphi_{n-1}(s), \qquad n \geq 1 \qquad (1)$$

$$\varphi_n(s) \triangleq \prod_{k=1}^{n}\frac{s-\overline{a_k}}{s+a_k}, \qquad n \geq 1$$

with $B_0(s) = \varphi_0(s) \equiv 1$. With respect to the usual inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(j\omega)\overline{g(j\omega)}\,\mathrm{d}\omega$$

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on $H_2(\Pi)$ these functions are orthonormal. Previous work on continuous time orthonormal bases has concentrated on special cases of the basis (1) wherein all the $\{a_k\}$ are the same real number $a_k = a \in \mathbf{R}$ in which case the ensuing basis is known as the 'Laguerre' basis [16, 15, 8, 23], or the case of all the $\{a_k, a_{k+1}\}$ being the same complex conjugate pair $a_k = a, a_{k+1} = \overline{a}$ [31].

An important motivation for the consideration of orthonormal parameterisations is for approximation purposes. In this setting, a dominant question must arise as to the quality of the approximation. Pertaining to this, one of the most fundamental properties that might be required is that linear combinations of the basis elements be capable of arbitrarily good approximation.

Put more precisely in the context of systems theory, this involves considering an element $f(s)$ living in a normed linear function space $(X, \|\cdot\|_X)$, and for arbitrary $\epsilon > 0$ and for sufficiently large n being able to find an element $g(s) \in \text{sp}\{B_k(s)\}_{k=1}^n$ such that $\|f - g\|_X \leq \epsilon$.

If this is in fact possible for arbitrarily small ϵ, then $\text{sp}\{B_k(s)\}_{k\geq 1}$ is said to be 'complete' in X. The choice of the function space depends on the application of the approximate model, but for quadratic optimal control purposes or mean square optimal prediction purposes, the choice $H_2(\Pi)$ would be appropriate, while for robust control purposes, the choices $A(\Pi)$ or $H_p(\Pi)$ (for large p) would be suitable.

With regard to the discrete-time basis defined on $\mathbf{D} \cup \mathbf{T}$ by

$$
\begin{aligned}
\mathcal{B}_n(z) &\triangleq \frac{\sqrt{1 - |\xi_n|^2}}{1 - \overline{\xi_n} z} \phi_{n-1}(z), \qquad (2) \\
\phi_n(z) &\triangleq \prod_{k=1}^{n} \frac{z - \xi_k}{1 - \overline{\xi_k} z}, \\
\phi_0(z) &\triangleq 1,
\end{aligned}
$$

the approximation issues have been addressed in [4] where the following result was obtained.

Theorem 2.1 *[4, Theorem 6 and Corollary 7], Consider the set of functions $\{\mathcal{B}_k(z)\}$ defined by (2). Then the set $X = \text{sp}\{\mathcal{B}_k(z)\}_{k\geq 0}$ is complete in $A(\mathbf{D})$ and $H_p(\mathbf{T})$ for all $1 \leq p < \infty$ if and only if the sequence of complex numbers $\xi_k \in \mathbf{D}$ satisfies*

$$
\sum_{k=1}^{\infty} (1 - |\xi_k|) = \infty. \qquad (3)
$$

The first result of this paper is, via to Theorem 3.1, to establish an analogous result for the continuous-time basis (1).

A function in $f(s) \in H_p(\Pi)$ is said to have 'asymptotic order' m if $f(s) = O(|s|^{-m})$ as $|s| \to \infty$. Clearly the bases defined by (1) have asymptotic order 1, but as illustrated in other work on continuous time orthonormal bases such as [31], for the purposes of model error approximation and minimisation, there is great utility in being able to construct bases of asymptotic order greater than 1. Accordingly, Theorem 4.1 in § 4 establishes a method to construct an infinite set of orthonormal bases, each of which have arbitrary asymptotic order, and whose linear span is norm dense in $H_p(\Pi)$ for all $1 \leq p < \infty$.

Up until and including § 4 the paper has established that approximants of arbitrary small H_p norm exist, but not what they might be. In § 5 a specific (and obvious) approximant is considered which is the generalised Fourier series approximant. There, via Theorem 5.4, it is established that this approximant is, in fact, of arbitrarily small H_p norm distance from the function being approximated for any $1 < p < \infty$. The paper concludes by showing how this continuous time result may be used to establish an equivalent discrete time one.

3 Complete Model Sets

The paper begins by presenting the following result

Theorem 3.1 *The model set spanned by the basis functions $\{B_n(s)\}_{n\geq 0}$ is complete in all of the spaces $H_p(\Pi)$, $1 \leq p < \infty$ and $A(\Pi)$ if and only if*

$$
\sum_{n=1}^{\infty} \frac{\text{Re}\{a_n\}}{1 + |a_n|^2} = \infty. \qquad (4)
$$

Proof. See [2] and [3]. ♣

3.1 Approximation of finite-dimensional systems

While the completeness result of Theorem 3.1 provides a theoretical pedigree for considering the bases (1) for system approximation purposes, it leaves open the question of the quality of approximation for a finite number of bases. Addressing this will consume the rest of this section, where a central tool is to use the so-called 'reproducing kernel' $K_n(s, \mu)$ associated with the linear space $\text{sp}\{B_k(s)\}_{k=1}^n$.

Lemma 3.2 *Consider the basis functions $\{B_k\}_{k=1}^n$ defined by (1). Then*

$$
K_n(s, \mu) \triangleq \sum_{k=1}^{n} \overline{B_k(\mu)} B_k(s) = \frac{1 - \overline{\varphi_n(\mu)} \varphi_n(s)}{s + \overline{\mu}}.
$$

Proof. See [2]. ♣

The utility of this result becomes apparent in the derivation of the following expression for the finite order approximation error.

Lemma 3.3 *Suppose $f(s)$ is analytic on Π and has a partial fraction expansion*

$$f(s) = \sum_{k=1}^{m} \frac{c_k}{s+\gamma_k}.$$

Define $f_n(s)$ as an approximation to $f(s)$ obtained by projection onto $\text{sp}\{B_k(s)\}_{k=1}^n$*:*

$$f_n(s) \triangleq \sum_{k=1}^{n} \langle f, B_k \rangle B_k(s).$$

Then

$$|f(j\omega) - f_n(j\omega)| \leq \sum_{k=1}^{m} \left| \frac{c_k}{j\omega + \gamma_k} \right| \prod_{\ell=1}^{n} \left| \frac{\gamma_k - a_\ell}{\gamma_k + \overline{a_\ell}} \right|. \quad (5)$$

Proof. See [2]. ♣

The result exposes the dependence of the approximation error on the choice of poles $\{a_n\}$ in the base $B_n(s)$. Namely, the closer the poles $\{a_n\}$ are chosen to the poles $\{\gamma_k\}$ of the function $f(s)$ being approximated then the more accurate the approximation of $f(s)$ will be, and in such a way as to decrease exponentially with increasing n.

Certainly the error bound (5) gives strong motivation for the consideration of the general basis (1), since (in contrast to the Laguerre and Kautz cases where all the poles are fixed at the same value) the increased flexibility of pole location $\{a_n\}$ will increase the possibility of making $|\gamma_k - a_\ell|$ small (for some ℓ) for every k, and hence making the total product $\prod_{\ell=1}^{n} |\gamma_k - a_\ell||\gamma_k + \overline{a_\ell}|^{-1}$ as small as possible.

4 Orthonormal Basis Functions with Prescribed Asymptotic Order

This section presents a derivation of model sets that are norm dense in $H_p(\Pi)$ for $1 < p < \infty$ and for which the orthonormal basis functions $B_n(s)$ defining the sets each have a prescribed asymptotic order. That is, $B_n(s) = O(|s|^{-m})$ as $|s| \to \infty$. The basis functions studied in the previous section all have asymptotic order $m = 1$.

The problem of synthesis of bases of arbitrary asymptotic order has been investigated in the literature for various specific cases of choice a_k of pole position or of asymptotic order m [28, 18, 7, 16].

In contrast to these previous specific cases available in the literature, the following result provides a recipe to construct bases of arbitrary asymptotic order m and with arbitrary pole position a_k (that satisfies (4)).

Theorem 4.1 *Suppose that (4) is satisfied. Let*

$$\begin{aligned} \psi_n(s) &\triangleq \frac{B_n(s)}{P(s)} - \sum_{k=1}^{n-1} \left\langle \frac{B_n}{P}, \psi_k \right\rangle \frac{\psi_k(s)}{\|\psi_k\|_2^2}, \\ \psi_1(s) &\triangleq \frac{B_0(s)}{P(s)} \end{aligned} \quad (6)$$

where $P(s)$ is an arbitrary mth order polynomial with roots in the complement of $\overline{\Pi}$ and $m > 1$. Then the basis functions

$$\widetilde{B}_n(s) \triangleq \frac{\psi_n(s)}{\|\psi_n\|_2}, \qquad n \geq 1 \quad (7)$$

are orthonormal and have the asymptotic order $\widetilde{B}_n(s) = O(|s|^{-m})$ as $|s| \to \infty$. Moreover $sp\{\widetilde{B}_n\}_{n\geq 1}$ is norm dense in $H_p(\Pi)$ for all $1 \leq p < \infty$.

Proof. See [3]. ♣

5 Convergence of Generalised Fourier Series in $H_p(\Pi)$

Let $\{B_k\}_{k\geq 1}$ be a set of basis functions which satisfy (4). Then $\{B_k\}_{k\geq 1}$ is a norm dense set of basis functions for $H_2(\Pi)$ and every $f \in H_2(\Pi)$ has a Fourier series expansion

$$\widehat{f}_n(s) \triangleq \sum_{k=1}^{n} \langle f, B_k \rangle B_k(s) \quad (8)$$

that converges to f in the $L_2(j\mathbf{R})$–norm. When $B_k = z^k$, $k = 1, 2, \cdots$ and the underlying space is $H_p(\mathbf{D})$, it is well known that every $f \in H_p(\mathbf{D})$ has a Fourier series which also converges in the $L_p(\mathbf{T})$–norm for all $1 < p < \infty$. In this section it is shown that the same is true for the basis functions in (1). First it is necessary to derive an upper bound on $|\varphi_n(s)|$.

Lemma 5.1 *Let φ_n be as in (1). Then for each $s \in \Pi$*

$$|\varphi_n(s)| \leq \exp\left(-\frac{1}{5} \left[1 - \left| \frac{1-s}{1+s} \right| \right] \sum_{k=1}^{n} \frac{Re\{a_k\}}{1+|a_k|^2} \right). \quad (9)$$

Proof. See [3]. ♣

Use of this result allows the calculation of the $L_p(j\mathbf{R})$ distance from a first order system to the model set spanned by B_k, $k = 1, 2, \cdots, n$.

Lemma 5.2 *Let $\widehat{f}_n$ be as in (8) and suppose that $f(s) = 1/(s+\gamma)$ is analytic on Π. Then*

$$\left\|f - \widehat{f}_n\right\|_p \leq \|f\|_p \, \exp\left(-\frac{2Re\{\gamma\}}{5|1+\gamma|^2} \sum_{k=1}^{n} \frac{Re\{a_k\}}{1+|a_k|^2}\right). \tag{10}$$

Proof. See [3]. ♣

Next it is established that the maps $f \mapsto \widehat{f}_n$ are bounded.

Lemma 5.3 *Let $\widehat{f}_n$ be as in (8). Then there exists a constant $C_p < \infty$, which depends only on p, such that for all $1 < p < \infty$*

$$\|\widehat{f}_n\|_p \leq (1 + C_p)\|f\|_p. \tag{11}$$

Proof. See [3]. ♣

For the precise value of C_p, the reader is referred to Chapter III in Garnett [11]. Using Lemmata 5.2–5.3, we prove the following main result of this section can now be established.

Theorem 5.4 *Consider the partial sums of the Fourier series defined by (8). Suppose that (4) holds. Then for all $1 < p < \infty$*

$$\lim_{n\to\infty} \|f - \widehat{f}_n\|_p = 0, \qquad f \in H_p(\Pi). \tag{12}$$

Proof. See [3]. ♣

The remainder of this section will be consumed with the extension of Theorem 5.4 to the discrete–time orthonormal basis functions (2).

The following is the discrete-time version of Lemma 5.2. In what follows, the notation $\|\cdot\|_p$ refers to the L_p–norms on the unit circle and the inner product for two functions $f, g \in H_2(\mathbf{T})$ is defined as

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{j\omega})\overline{g(e^{j\omega})}\,\mathrm{d}\omega.$$

Lemma 5.5 *Define $\widehat{f}_n$ as an approximation to $f \in H_1(\mathbf{D})$, obtained by projection onto $\{\mathcal{B}_k\}_{k=1}^n$:*

$$\widehat{f}_n(z) = \sum_{k=1}^{n}\langle f, \mathcal{B}_k\rangle \mathcal{B}_k(z). \tag{13}$$

Suppose f is analytic and magnitude bounded by K on the disk $\{z : |z| < R\}$ for some $R > 1$. Then

$$\|f - \widehat{f}_n\|_\infty \leq \frac{KR}{R-1}\exp\left(-\frac{R-1}{2R}\sum_{k=1}^{n}(1 - |\xi_k|)\right). \tag{14}$$

Proof. See Lemma 4 in [4]. ♣

Lemma 5.6 *Let $\widehat{f}_n$ be as in (13). Then there exists a constant $C_p < \infty$, which depends only on p, such that for all $1 < p < \infty$*

$$\|\widehat{f}_n\|_p \leq (1 + C_p)\|f\|_p. \tag{15}$$

Proof. See [4]. ♣

Theorem 5.7 *Consider the partial sums of the Fourier series defined by (13). Suppose that (3) holds. Then for all $1 < p < \infty$*

$$\lim_{n\to\infty} \|f - \widehat{f}_n\|_p = 0, \qquad f \in H_p(\mathbf{D}). \tag{16}$$

Proof. See [4]. ♣

The cases $H_1(\mathbf{D})$ and $A(\mathbf{D})$ can not be included in Theorem 5.7. For example when $B_k = z^k$, $k = 1, 2, \cdots$, it is well known that every integrable function does not necessarily admit a Fourier series converging in any of these spaces.

6 Conclusions

This paper has provided analysis of the approximation properties of certain general classes of rational orthonormal basis functions. The nature of the results was such as to establish that for continuous time linear time invariant system modelling, arbitrarily small H_p norm approximation error was possible for any $p \in [1, \infty)$ and furthermore, for the case of $p \in (1, \infty)$, this may be provided while at the same time using bases with arbitrary asymptotic order. Finally, a specific construction of the system approximant via Fourier decomposition was shown to be one in which the H_p norm error is arbitrarily small for any $p \in (1, \infty)$. The results have application in the analysis and design of robust estimation and control strategies.

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