On copositive Lyapunov functions for a class of monotone systems

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***Abstract*— This paper considers several explicit formulas for the construction of copositive Lyapunov functions for global asymptotic stability with respect to monotone systems evolving in either discrete or continuous time. Such monotone systems arise as comparison systems in the study of interconnected large-scale nominal systems. A copositive Lyapunov function for such a comparison system can then serve as prototype Lyapunov functions for the nominal system. We discuss several constructions from the literature in a unified framework and provide sufficiency criteria for the existence of such constructions.**

I. Introduction

We consider autonomous continuous- and discrete-time dynamical systems evolving in $\mathbb{R}^n_+$ that have an order preserving flow, i.e., $x \prec y$ implies $\phi(t,x) \prec \phi(t,y)$ for all times t where both trajectories exist. Here $\prec$ stands for any of the relations $\leq, <, \ll$, which denote the partial order induced by the cone $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0 \forall i\}$. In particular, $x \leq y$ if $x_i \leq y_i$ for all i; $x < y$ if [$x \leq y$ and $x \neq 0$]; and $x \ll y$ if $x_i < y_i$ for all i.

The systems under consideration take the form

$$\dot{x} = f(x) \qquad (\Sigma_C)$$

with $f\colon \mathbb{R}^n_+ \to \mathbb{R}^n$ locally Lipschitz in the continuous-time case and

$$x^+ = g(x) \qquad (\Sigma_D)$$

with $g\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ continuous in the discrete-time case. Here (Σ_D) is short for $x(k+1) = g(x(k))$. Throughout the paper we will refer simply to Σ to denote both systems (Σ_C),(Σ_D) simultaneously.

Under the assumption that the origin is an equilibrium for Σ it is well known that the positive orthant $\mathbb{R}^n_+$ is invariant if and only if f is *quasi-monotone nondecreasing* and g is *monotone*, i.e., if for all $x, y \in \mathbb{R}^n_+$,

$$x \leq y \text{ and } x_i = y_i \implies f_i(x) \leq f_i(y)$$

and, respectively,

$$x \leq y \implies g(x) \leq g(y).$$

If f is differentiable then it is quasi-monotone nondecreasing if and only if $\frac{\partial f_i}{\partial x_j}(x) \geq 0$ for all $i \neq j$ and all $x \in \mathbb{R}^n_+$. In what follows, we take as a standing assumption that f is quasi-monotone nondecreasing and g is monotone.

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Assuming that the origin is globally asymptotically stable it is well known that there must exist a Lyapunov function V such that for some $\mathcal{K}_\infty$ functions ψ_1, ψ_2,

$$\psi_1(\|x\|) \leq V(x) \leq \psi_2(\|x\|) \qquad (1)$$

and there exists a continuous positive definite function $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $x > 0$,

$$D^+_\Sigma V(x) \leq -\alpha(\|x\|).$$

Here D^+_Σ denotes either the upper right Dini derivative

$$\limsup_{h\to 0^+} \frac{V(x + hf(x)) - V(x)}{h} \qquad (2)$$

in case of (Σ_C) and

$$V(g(x)) - V(x)$$

in case of (Σ_D), cf. [20] and [12].

Remark I.1 *Recall that Rademacher's Theorem states that, if V is locally Lipschitz, then*

$$D^+_{\Sigma_C} V(x) = \langle \nabla V(x), f(x) \rangle$$

almost everywhere. In the case where V is differentiable, the above is in fact satisfied for all $x \in \mathbb{R}^n$.

Here we are interested in Lyapunov functions that take a special form, namely

$$V(x) = \max_i \rho_i(x_i) \qquad (3)$$

and

$$W_C(x) = \sum_i \int_0^{x_i} \lambda_i(s) \mathrm{d}s \qquad (4)$$

$$W_D(x) = \sum_i \lambda_i(x_i) x_i = \langle \lambda(x), x \rangle, \qquad (5)$$

where the functions ρ_i are of class $\mathcal{K}_\infty$ and the $\lambda_i\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfy

$$\lambda_i(s) > 0 \quad \forall s > 0$$
$$\int_0^\infty \lambda_i(s) \mathrm{d}s = \infty, \qquad (6)$$

and $\lambda(s) = (\lambda_1(s_1), \ldots, \lambda_n(s_n))^T$.

Such Lyapunov functions are of particular interest when the functions ρ_i and λ_i can be computed explicitly. This in turn is of interest especially when (large-scale) interconnected nonlinear systems are considered. Here a monotone system Σ usually arises in form of a comparison system [5], [10], [8], [9], [17], [18].

Monotone systems also arise in other contexts such as systems biology [2], ecology models describing population dynamics, and economic models [19]. More general chemical reaction models are frequently modeled as cooperative monotone systems that sometimes also possess homogeneity properties [1]. Homogeneous systems are in some sense special among these, as they remain stable under arbitrary delay [14]. In this paper we will make no homogeneity assumptions.

The paper is organized as follows. In section II we introduce the $\not\geq$-condition, which is necessary for global asymptotic stability of the origin and also discuss sufficiency results. In section III we provide several Lyapunov-type results which mostly rely on the existence of certain scaling functions. The existence of such functions is discussed in section IV and further types of Lyapunov functions are introduced. Finally, in section V we briefly outline one of several ways how the results in this paper can be utilized to construct Lyapunov functions for arbitrary large-scale systems.

II. Necessary and sufficient conditions for asymptotic stability

Here we collect a few known results regarding asymptotic stability of the origin with respect to system Σ, cf. [18], [16] for proofs.

With a slight abuse of notation, we denote the flow of system Σ_C by $\phi\colon \mathbb{R}_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$ and the flow of system Σ_D by $\phi\colon \mathbb{Z}_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$. Depending on the context, we will use the time variable t to denote either a non-negative real number or a non-negative integer.

Definition II.1 *The origin is* stable in the sense of Lyapunov, *if for all $\epsilon > 0$ there exists a $\delta > 0$ such that any solution starting at $x \in \mathbb{R}^n_+$ with $\|x\| < \delta$ exists for all positive times and satisfies $\|\phi(t,x)\| < \epsilon$ for all $t \geq 0$.*

The origin is attractive *with* domain of attraction $\mathcal{B}$ *if for every $x \in \mathcal{B}$, $\lim_{t\to\infty} \phi(t,x) = 0$. If $\mathcal{B} = \mathbb{R}^n_+$ then we say that the origin is* globally asymptotically stable *(GAS) with respect to system Σ.*

Observe that due to monotonicity of the flow of Σ, $x \in \mathcal{B}$ always implies $y \in \mathcal{B}$ if $y \leq x$.

Lemma II.2 *Assume that the origin is attractive with respect to Σ with domain of attraction $\mathcal{B}$. Then $f(x) \not\geq 0$ for all $x \in \mathcal{B}$ and $g(x) \not\geq x$ for all $x \in \mathcal{B}$.*

The proof follows easily by contradiction. This simple result has an important consequence as we shall see next.

Definition II.3 *We say that Σ satisfies the* $\not\geq$-condition *if, respectively, $f(x) \not\geq 0$ and $g(x) \not\geq x$ for all $x > 0$.*

Definition II.4 *Denote the* decay set $\Omega \subset \mathbb{R}^n_+$ *with respect to Σ by*

$$\Omega_C = \{x \in \mathbb{R}^n_+ \colon f(x) \ll 0\},$$

and, respectively,

$$\Omega_D = \{x \in \mathbb{R}^n_+ \colon g(x) \ll x\}.$$

By S_r we denote the sphere with respect to the 1-norm of radius r in $\mathbb{R}^n_+$, i.e., the set

$$S_r = \Big\{x \in \mathbb{R}^n_+ \colon \sum_i x_i = r\Big\}. \tag{7}$$

Theorem II.5 *Assume that the origin is attractive with respect to Σ with a domain of attraction $\mathcal{B}$. For all $r > 0$ such that $S_r \subset \mathcal{B}$, the intersection $S_r \cap \Omega$ is nonempty.*

They key to the importance of this result lies in the fact that Ω is invariant under Σ, i.e., $x \in \Omega$ implies $\phi(t,x) \in \Omega$ for all times $t \geq 0$.

Corollary II.6 *If the origin is attractive with respect to Σ then it is also stable.*

This result is a special feature of monotone systems on $\mathbb{R}^n_+$. Equilibria of general nonlinear systems can be attractive and yet unstable, cf. the example in [7, §40, pp191–194].

Definition II.7 *We say that a set $A \subset \mathbb{R}^n_+$ is* jointly unbounded *if for every $x \in \mathbb{R}^n_+$ there exists a $y \in A$ with $y \geq x$.*

Here *jointly* refers to the fact that a jointly unbounded set is not only not bounded, but it is also unbounded in every coordinate direction.

Proposition II.8 *Assume that Σ satisfies the $\not\geq$-condition and that Ω is jointly unbounded. Then the origin is globally asymptotically stable.*

It is known for particular classes of maps f and g that if Σ satisfies the $\not\geq$-condition and Ω is jointly unbounded then a path can be constructed in $\mathbb{R}^n_+$, which is component-wise the image of $\mathcal{K}_\infty$ functions, i.e., that there exist functions $\sigma_i \in \mathcal{K}_\infty$ such that with $\sigma(r) = (\sigma_1(r), \ldots, \sigma_n(r))^T$, one has $\sigma(r) \in \Omega$ for all $r > 0$. These particular classes of maps include homogeneous maps [1], as well as maps induced by matrices of nonlinear functions [5], [16], of which linear maps are a special case.

Definition II.9 *We call an* Ω-path *a map $\sigma\colon \mathbb{R}_+ \to \mathbb{R}^n_+$ such that $\sigma(r) \in \Omega$ for all $r > 0$.*

Such paths will become useful in the next section for the construction of Lyapunov functions of the form (3).

We also state the following proposition which extends the known discrete-time result to the continuous-time domain as well as Lemma II.2.

Proposition II.10 *Assume Σ satisfies the $\not\geq$-condition. Then $\phi(t,x) \not\geq x$ for all $t > 0$ and all $x > 0$.*

Proof: Assume the claim is false then there exists a $t^* > 0$ and an $x > 0$ such that $\phi(t^*,x) \geq x$.

In the discrete-time case let $z := \max_{k=0,\dots,t^*-1} \phi(k,x)$ where the maximum is taken component-wise; i.e., $z_i := \max_{k=0,\dots,t^*-1} \phi_i(k,x)$ and $z := [z_1 \dots z_n]^T$. We then have

$$\begin{aligned} g(z) &\geq \max_{k=1,\dots,t^*} \phi(k,x) && \text{by monotonicity of } g \\ &= \max_{k=0,\dots,t^*} \phi(k,x) && \text{since } \phi(t^*,x) \geq x \\ &\geq \max_{k=0,\dots,t^*-1} \phi(k,x) = z, \end{aligned}$$

which poses a contradiction to the $\ngeq$-condition.

In the continuous-time case we take instead $z := \sup_{\tau\in[0,t^*]} \phi(\tau,x)$, where the supremum is taken component-wise, and observe that

$$\phi(t^*+\delta,x) \geq \phi(\delta,x) \tag{8}$$

for arbitrary $\delta > 0$ by monotonicity of ϕ. Now we compute for arbitrary $\epsilon > 0$,

$$\begin{aligned} \sup_{\delta\leq\epsilon} \phi(\delta,z) &\geq \sup_{\tau\in[\epsilon,t^*+\epsilon]} \phi(\tau,x) && \text{by monotonicity of } \phi \\ &\geq \sup_{\tau\in[0,t^*+\epsilon]} \phi(\tau,x) && \text{by (8)} \\ &\geq \sup_{\tau\in[0,t^*]} \phi(\tau,x) = z. \end{aligned}$$

By the fundamental theorem of calculus we have for arbitrary small $\delta > 0$ that

$$\phi(\delta,z) - z = \int_0^\delta f(\phi(s,z))ds \geq 0. \tag{9}$$

Since δ is arbitrarily small, by the $\ngeq$-condition and continuity of f there exists an index i such that

$$f_i(\phi(s,z)) < 0$$

for all $s \in [0,\delta]$. But this contradicts (9) which imposes $f_i(\phi(s,z)) \geq 0$ for all $s \in [0,\delta]$.

Hence in both time domains we yield a contradiction, proving that indeed $\phi(t,x) \ngeq x$ for all $x > 0$ and $t > 0$. ■

Remark II.11 *A consequence of the proof of Proposition II.10 is that we can turn any continuous-time system satisfying the $\ngeq$-condition into a discrete-time system by taking*

$$g(x) := \phi_C(\epsilon,x)$$

for some fixed $\epsilon > 0$, where ϕ_C denotes the flow with respect to (Σ_C). Interestingly, ϵ does not even have to be small and yet the sampled system satisfies the $\ngeq$ condition.

III. Global Lyapunov functions

Proposition III.1 *Consider system Σ. Let $V\colon \mathbb{R}^n_+ \to \mathbb{R}_+$ be a continuous function satisfying (1) for some $\psi_1,\psi_2 \in \mathcal{K}_\infty$ and let $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous positive definite function. Assume that*

$$D^+_\Sigma V(x) \leq -\alpha(\|x\|)$$

for all $x \geq 0$. Then the origin is globally asymptotically stable. Moreover, there exists a smooth Lyapunov function $\tilde V$ and a class $\mathcal{K}_\infty$-function $\tilde\alpha$ such that $D^+V(x) \leq -\tilde\alpha(\|x\|)$ for all $x > 0$.

This result is well known in Lyapunov stability theory, see [12] for the discrete-time case and [21, Thm. 8.3, p.32] or [15, Cor. II.6.4, p.90] (which assumes α to be of class $\mathcal{K}$) together with [20] (for a general smooth converse result) for the continuous-time case.

Now we can state and prove the main results on existence of global Lyapunov functions of the form (3),(4),(5).

Theorem III.2 *Assume that there exists an Ω-path σ with respect to Σ with $\sigma_i \in \mathcal{K}_\infty$ for all i. In case of (Σ_C) assume in addition that for all i, σ_i satisfies for all $r > 0$*

$$\limsup_{h\to0^+} \frac{\sigma_i^{-1}(\sigma_i(r)-h)-r}{h} > 0. \tag{10}$$

Then (3) defines a non-smooth Lyapunov function for Σ with the choice $\rho_i = \sigma_i^{-1}$.

It has to be noted that V cannot be smooth, even if the functions σ_i were smooth due to the maximization.

Proof: Using the equivalence of norms on $\mathbb{R}^n$ the verification of (1) is plain since the functions σ_i are of class $\mathcal{K}_\infty$.

Continuous-time: We define the auxiliary functions

$$\beta_i(r) := \limsup_{h\to0^+} \frac{\sigma_i^{-1}(\sigma_i(r)-h)-r}{h}$$

and

$$\tau_i(r) = -f_i(\sigma(r)).$$

Note that both $\beta_i(r)$ and $\tau_i(r)$ are positive for $r > 0$ by (10) and the fact that σ is an Ω-path.

We first assume that the maximum in (3) is attained for a unique i^*. Denote $r = \sigma_{i^*}^{-1}(x_{i^*}) = V(x)$. We compute

$$\begin{aligned} D^+_\Sigma V(x) &= \limsup_{h\to0^+} \frac{\sigma_{i^*}^{-1}(x_{i^*}+hf_{i^*}(x)) - \sigma_{i^*}^{-1}(x_{i^*})}{h} \\ &= \limsup_{h\to0^+} \frac{\sigma_{i^*}^{-1}(\sigma_{i^*}(r)+hf_{i^*}(x)) - r}{h} \end{aligned}$$

since $x \leq \sigma(r)$ this is

$$\begin{aligned} &\leq \limsup_{h\to0^+} \frac{\sigma_{i^*}^{-1}(\sigma_{i^*}(r)+hf_{i^*}(\sigma(r))) - r}{h} \\ &\leq \limsup_{h\to0^+} \frac{\sigma_{i^*}^{-1}(\sigma_{i^*}(r)-h) - r}{h} f_{i^*}(\sigma(r)) \\ &\leq -\beta_i(r)\tau_i(r) \\ &\leq -\alpha(r) = -\alpha(V(x)) < 0, \end{aligned}$$

where α is any continuous positive definite function satisfying $\alpha(r) \leq \min_i \beta_i(r)\tau_i(r)$ for all $r > 0$.

If the maximum is not unique in (3) then it suffices to repeat the above calculation for

$$\limsup_{h\to0^+} \frac{\sigma_i^{-1}(x_i+hf_i(x)) - \sigma_i^{-1}(x_i)}{h}$$

for every i such that $\sigma_i^{-1}(x_i) = V(x)$ to obtain that

$$\limsup_{h\to 0^+} \frac{\sigma_i^{-1}(x_i + hf_i(x)) - \sigma_i^{-1}(x_i)}{h} < -\alpha(V(x)).$$

Now note that this implies $D_\Sigma^+ V(x) \le -\alpha(V(x))$ by [15, Thm. A1.2.7, p.349].

Discrete-time: In the discrete-time case we have to consider the difference $V(g(x)) - V(x)$. Assume that $x > 0$. Denote by $r_{g(x)}$ the smallest $r \ge 0$ such that $g(x) \le \sigma(r)$, and by r_x the smallest $r > 0$ such that $x \le \sigma(r)$. In other words $r_{g(x)} = V(g(x))$ and $r_x = V(x)$.

By monotonicity of g we have $g(x) \le g(\sigma(r_x)) \ll \sigma(r_x)$. Now apply V to this inequality to obtain $V(g(x)) \le V(g(\sigma(r_x))) < V(\sigma(r_x)) = V(x)$. Hence $V(g(x)) - V(x) \le V(g(\sigma(V(x)))) - V(x) := -\alpha(V(x)) < 0$ and the function α is positive definite and continuous.

This concludes the proof. ∎

It should be noted that this kind of Lyapunov function has been introduced in the context of interconnected nonlinear systems in the case $n = 2$ interconnected systems in [11] and for general n in [5]. Both approaches made use essentially of the continuous-time variant of this result.

Theorem III.3 *Assume there exist functions $\lambda_i \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

- $\lambda_i(s) > 0$ *for all* $s > 0$,
- $\int_0^\infty \lambda_i(s)\mathrm{d}s = \infty$, *and*
- *for all* $x > 0$,

$$\lambda^T(x) f(x) < 0 \qquad \textit{and, respectively,} \qquad (11)$$

$$\lambda^T(g(x))g(x) < \lambda^T(x)x. \qquad (12)$$

Then the function W given by (4)/(5) *is a Lyapunov function for Σ. In the continuous-time case W is differentiable and it is continuously differentiable if the functions λ_i are continuous. In the discrete-time case W has the same smoothness properties as the functions λ_i.*

Proof: The first two assumptions on the functions λ_i guarantee the existence of functions $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that (1) holds.

In the continuous-time case observe that W is by construction differentiable. We compute $D^+W(x) = \langle \nabla W(x), f(x)\rangle = \langle \lambda(x), f(x)\rangle < 0$ for all $x > 0$ by assumption.

In the discrete-time case we have $D^+W(x) = W(g(x)) - W(x) = \lambda^T(g(x))g(x) - \lambda^T(x)x < 0$ by assumption.

The set S_r defined in (7) is compact, so $\tilde{\alpha}(r) := \max_{x\in S_r} -D^+W(x) > 0$ for all $r > 0$. Now it is plain to bound $\tilde{\alpha}$ from below by a continuous positive definite function α which satisfies $D^+W(x) < -\alpha(\|x\|_1)$. By the equivalence of norms on $\mathbb{R}^n$ we could as well find a function $\overline{\alpha}$ such that $D^+W(x) < -\overline{\alpha}(\|x\|)$.

The smoothness properties of W are obvious consequences of the respective properties of λ_i and, in the continuous-time case, of the integration. ∎

IV. Existence of the scaling functions

In this section we provide a few sufficiency results on the existence of the scaling functions σ_i and λ_i which are required in the global constructions of Theorems III.2 and III.3.

A. The linear case

The linear case is completely solved and the problem is essentially to compute suitable eigenvectors. We briefly recall the facts.

Recall that $M = (m_{ij}) \in \mathbb{R}^{n\times n}$ is called a Metzler matrix if $m_{ij} \ge 0$ for all $i \ne j$. The map $f(x) = Mx$ yields the linear prototype of a monotone system (Σ_C). A necessary and sufficient criterion for global asymptotic stability with respect to (Σ_C) is that the spectral abscissa

$$a(M) := \max\{\mathrm{Re}\lambda \colon \lambda \text{ is an eigenvalue of } M\}$$

satisfies $a(M) < 0$. In this case one has $P := M + a(M)I \in \mathbb{R}_+^{n\times n}$ and $a(M) - \rho(P) < 0$.

Therefore, if Σ is linear then there exists a non-negative matrix $P \in \mathbb{R}_+^{n\times n}$ and a number $a > 0$ such that Σ_C is given by $f(x) = (-aI + P)x$ or Σ_D is given by $g(x) = Px$. Global asymptotic stability of the origin is equivalent to the condition that

$$\rho(P) < a \qquad (13)$$

in the continuous-time case and

$$\rho(P) < 1 \qquad (14)$$

in the discrete-time case. Both (13) and (14) are equivalent to the respective $\not\ge$-condition. So for stability of monotone linear systems this condition is not only necessary but also sufficient.

If P is irreducible (cf. [3]) there exist unique left and right eigenvectors $l, s \in \mathbb{R}_+^n$, $l, s \gg 0$, such that $Ps = \rho(P)s$ and $l^T P = \rho(P)l^T$. If P is reducible there still exist vectors $l, s \gg 0$, but they are not unique. They satisfy in the discrete-time case $Ps \ll s$ and $l^T P \ll l^T$ and in the continuous-time case $Ps \ll as$ and $l^T P \ll al^T$. In all cases, the mappings $r \mapsto rl$ and $r \mapsto rs$ yield suitable vector-valued functions $\lambda(r)$ and $\sigma(r)$, respectively [16].

Since $\rho(P) = \rho(P^T)$ for every matrix P, the left- and right-eigenvector existence problems are dual to each other, and one can be solved if and only if the other can be solved.

B. The nonlinear case

1) Scaling functions for the max-type Lyapunov function constructions: Here we consider the existence of an Ω-path $\sigma \colon \mathbb{R}_+ \to \mathbb{R}_+^n$ which can be used as candidate scaling functions in Theorem III.2.

Definition IV.1 *We say that system Σ is given in terms of $\mathcal{K}_\infty$-gain matrices if*

- *in the continuous-time case there exist functions* $\alpha_i \in \mathcal{K}_\infty$ *and* $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ *such that*

$$f(x) = \begin{pmatrix} -\alpha_1(x_1) + \sum_{j=1}^n \gamma_{1j}(x_j) \\ \vdots \\ -\alpha_n(x_n) + \sum_{j=1}^n \gamma_{nj}(x_j) \end{pmatrix} ;\ \textit{and} \qquad (15)$$

- *in the discrete-time case there exist functions* $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ *such that*

$$g(x) = \begin{pmatrix} \sum_{j=1}^n \gamma_{1j}(x_j) \\ \vdots \\ \sum_{j=1}^n \gamma_{nj}(x_j) \end{pmatrix}. \qquad (16)$$

In both cases we define $\Gamma = (\gamma_{ij})$*, a matrix whose entries are nonlinear functions or zero. In the continuous-time case we also define* $A = \mathrm{diag}(\alpha_i)$*. We say that a matrix* $\Gamma = (\gamma_q ij)$ *with* $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ *is irreducible, if the matrix* $G = (g_{ij})$ *with*

$$g_{ij} = \begin{cases} 1 & \textit{if } \gamma_{ij} \in \mathcal{K}_\infty \\ 0 & \textit{otherwise} \end{cases}$$

is irreducible, cf. [3]. A matrix is reducible if it is not irreducible.

The motivation behind the naming "$\mathcal{K}_\infty$-gain matrices" is that f and g can be written as $f(x) = -A(x) + \Gamma(x)$ and, respectively, $g(x) = \Gamma(x)$, when we associate to gain matrices $A = \mathrm{diag}(\alpha_i)$ and $\Gamma = (\gamma_{ij})$ a notion of matrix-vector-application (instead of matrix-vector-multiplication) as in (20)–(21). See [4], [5] for more general definitions.

Theorem IV.2 *If* Σ *is given in terms of* $\mathcal{K}_\infty$*-gain matrices with irreducible matrix* Γ *and* Σ *satisfies the* $\not\geq$*-condition then there exits an* Ω*-path* $\sigma\colon \mathbb{R}_+ \to \mathbb{R}^n_+$ *which is piecewise linear on* $(0,\infty)$ *and every component function is of class* $\mathcal{K}_\infty$.

The theorem asserts that in particular every component function σ_i is of class $\mathcal{K}_\infty$ and piecewise linear, which implies that (10).

Proof: This result has been proven for the discrete-time case e.g. in [16]. The continuous-time case reduces to the discrete-time case by the observation that if $f(x) = -A(x) + \Gamma(x)$ is of the form (20) and satisfies $f(x) \not\geq 0$ for all $x > 0$ then also $\Gamma(x) \not\geq A(x)$ and also $(\Gamma \circ A^{-1})(x) \not\geq x$ for all $x > 0$. Now observe that $\Gamma \circ A^{-1}$ corresponds to a matrix with entries $\gamma_{ij} \circ \alpha_j \in \mathcal{K}_\infty \cup \{0\}$. ∎

A reducible matrix $\Gamma = (\gamma_{ij})$ can be, by virtue of a renumbering of the coordinates, brought into the form

$$\Gamma = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & \Gamma_{22} \end{pmatrix}, \qquad (17)$$

where Γ_{ij} are again gain matrices. Both Γ_{11} and Γ_{22} are square and Γ_{22} is either irreducible or a 1×1-zero block, cf. [3], [16]. In the following result we apply the same partition induced by (17) also the vector $x = (x_1^T, x_2^T)^T \in \mathbb{R}^n_+$ with $n = n_1 + n_2$ and to the gain matrix $A = diag(A_1, A_2)$.

Theorem IV.3 *Assume that* Σ *is given in terms of* $\mathcal{K}_\infty$*-gain matrices with such that* Γ *is of the form* (17)*. Assume* Σ *satisfies the* $\not\geq$*-condition and that there exists a* $\mathcal{K}_\infty$ *function* κ *such that with* $D = \mathrm{diag}(\mathrm{id} + \kappa)$*:*

- *In the continuous-time case*

$$(\Gamma_{11} \circ A_1^{-1} \circ D)(x_1) \not\geq x_1 \textit{ for all } x_1 > 0 \qquad (18)$$

or equivalently

$$-A_1(x_1) + (D \circ \Gamma_{11})(x_1) \not\geq 0 \textit{ for all } x_1 > 0.$$

- *In the discrete-time case*

$$(\Gamma_{11} \circ D)(x_1) \not\geq x_1 \textit{ for all } x_1 > 0. \qquad (19)$$

Then there exists a σ*-path* $\sigma\colon \mathbb{R}_+ \to \mathbb{R}^n_+$ *with* $\mathcal{K}_\infty$ *component functions, and whose first* n_1 *component functions are piecewise linear on* $(0,\infty)$ *and whose second* n_2 *component functions satisfy* (10).

Proof: By Theorem IV.2 there exits a Ω-path $\sigma_2\colon \mathbb{R}_+ \to \mathbb{R}^{n_2}_+$ with $\mathcal{K}_\infty$ component functions for the subsystem $\dot{x} = -A_2(x_2) + \Gamma_{22}(x_2)$ and, respectively, $x_2^+ = \Gamma_{22}(x_2)$, which is piecewise linear on $(0,\infty)$.

The equivalence between the two assumptions in the continuous-time case follows from a short calculation.

Now we consider the discrete-time case. It follows from the assumptions that by [16, Thm.5.10] there exists another Ω-path $\sigma_1\colon \mathbb{R}_+ \to \mathbb{R}^{n_1}_+$ with $\mathcal{K}_\infty$ component functions for the subsystem $x_1^+ = \Gamma_{11}(x_1)$, which is piecewise linear on $(0,\infty)$. Furthermore, this path σ_1 has the stronger property that there exists a $\mathcal{K}_\infty$ function ξ such that with $\tilde{D} = \mathrm{diag}(\mathrm{id} + \xi)$,

$$(\Gamma_{11} \circ \tilde{D})(\sigma_1(r)) \ll \sigma_1(r).$$

for all $r > 0$. By [16, Lemma 2.4] there exists a function $\eta \in \mathcal{K}_\infty$ such that $(\mathrm{id}+\kappa)^{-1} = \mathrm{id} - \eta$. Writing $\overline{D} = \mathrm{diag}(\eta)$ the last inequality is equivalent to

$$\Gamma_{11}(\sigma_1(r)) \ll \sigma_1(r) - \overline{D}(\sigma_1(r)) \text{ for all } r > 0.$$

Observe that it is possible to find a function $\mu \in \mathcal{K}_\infty$ with a $\mathcal{C}^1$ inverse such that

$$\overline{D}(\sigma_1(r)) > \Gamma_{12}(\sigma_2(\mu(r)))$$

for all $r > 0$. Also observe that

$$\Gamma_{22}(\sigma_2(\mu(r))) \ll \sigma_2(\mu(r))$$

for all $r > 0$. Now define

$$\sigma(r) = \begin{pmatrix} \sigma_1(r) \\ \sigma_2(\mu(r)) \end{pmatrix}$$

and observe that all component functions σ^i of σ satisfy (10) by construction. The existence of σ for the continuous-time case follows from the discrete-time case using the same argument applied to $\Gamma \circ A^{-1}$ as in the proof of Theorem IV.2. ∎

Remark IV.4 *Theorem IV.3 uses a weaker assumption that the related result [16, Thm.5.10]. Instead of requiring the* *monotone map D to strengthen the $\not\geq$-condition for all n coordinates, our result only requires the stronger assumption for the first n_1 coordinates.*

It should be noted that related existence results can also be found in [9], [6].

2) Scaling functions for the sum-type Lyapunov function constructions: We start with a negative example.

Example IV.5 *Consider $g\colon \mathbb{R}^2_+ \to \mathbb{R}^2_+$ given by*

$$g(x) = \begin{bmatrix} 1/2 \cdot x_2^2 \\ \sqrt{x_1} \end{bmatrix} \not\geq x$$

for all $x > 0$. It can be verified that there exists no fixed vector $\lambda \in \mathbb{R}^2_+$, $\lambda \gg 0$, such that $V(x) = \lambda^T x$ is a Lyapunov function for the discrete-time dynamics induced by g, i.e., that $V(g(x)) - V(x) < 0$ for all $x \in \mathbb{R}^2_+$, $x \neq 0$. To see this, it is sufficient to try to compute a vector λ, say $\lambda = (\lambda_1, 1)^T$, so that

$$\lambda^T[g(x) - x] < 0, \qquad \textit{for all } x \in \mathbb{R}^2_+, \|x\|_1 = 10.$$

As it can be seen after some calculation, this is impossible. So unlike for linear systems, a copositive Lyapunov function cannot be chosen to be linear in this nonlinear example.

In the continuous-time case, assume a non-negative matrix $P \in \mathbb{R}^{n\times n}_+$ and a number $a > 0$ are given satisfying (13) and that $h\colon \mathbb{R}^n \to \mathbb{R}^n$ is a monotone diffeomorphism with $h(0) = 0$ that leaves $\mathbb{R}^n_+$ invariant, i.e.,

$$h(\mathbb{R}^n_+) = \mathbb{R}^n_+.$$

Furthermore, assume that f is of the special form

$$f(x) = Jh\left(h^{-1}(x)\right) M h^{-1}(x) \tag{20}$$

where $Jh\left(h^{-1}(x)\right)$ denotes the Jacobian of $h(\cdot)$ evaluated at $h^{-1}(x)$.

Proposition IV.6 *Let P, $a > 0$, and h be given as above with $\rho(P) < a$. Let f be of the form* (20). *Then there exists a positive vector $l \gg 0$ such that $W(x) = l^T h^{-1}(x)$ is a Lyapunov function for system* (Σ_C).

Note that this time W is of the form (4) if and only if h acts component wise on the positive orthant, i.e., there exist functions η_i such that $h(x) = (\eta_1(x_1), \ldots, \eta_n(x_n))^T$. In this case the functions η_i^{-1} must be of class $\mathcal{K}_\infty$ and differentiable, and then we have

$$W(x) = \sum_i \int_0^{x_i} l_i\, \eta_i^{-1}(s)\mathrm{d}s.$$

Proof: The existence of the vector $l \gg 0$ satisfying $l^T M x < \frac{1}{2}(-a + \rho(P)) l^T x$ follows as in the previous subsection by standard results, see again [16]. Again it is clear that an estimate of the form (1) must hold for V.

We compute

$$\begin{aligned} D^+W(x) &= \langle \nabla W(x), f(x)\rangle \\ &= l^T Jh^{-1}(x)\, Jh\left(h^{-1}(x)\right) M h^{-1}(x) \\ &= l^T M h^{-1}(x) \\ &< \frac{1}{2}(-a + \rho(P)) l^T h^{-1}(x) \\ &= -\alpha W(x) \end{aligned}$$

with $\alpha = (a - \rho(P))/2 > 0$. ■

In the discrete-time case, we assume a non-negative matrix $P \in \mathbb{R}^{n\times n}_+$ is given satisfying (14). Furthermore, assume there exists a monotone homeomorphism $h\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ such that g takes the special form

$$g(x) = h^{-1}(Ph(x)). \tag{21}$$

Proposition IV.7 (discrete-time case) *Let h and P be as above with $\rho(P) < 1$. Let g be of the form* (21). *Then there exists a positive vector $l \gg 0$ such that $V(x) = l^T h(x)$ is a Lyapunov function for system* (Σ_D).

Note that V is not of the form (5), even if h acts component-wise. Observe further that in the example above g can be written as

$$g(x) = h^{-1}\big(Ph(x)\big)$$

with $h(x) = (x_1, x_2^2)^T$ and $P = \begin{bmatrix} 0 & 1/2 \\ 1 & 0 \end{bmatrix}$. Observing that $\lambda^T = (1, 3/4)$ satisfies $\lambda^T P = (3/4, 1/2) \ll \lambda^T$, it follows by the proposition that a Lyapunov function can be taken to be

$$V(x) = x_1 + \frac{3}{4}x_2^2.$$

Proof: The existence of the vector $l \gg 0$ follows as in the previous subsection by standard results, see, e.g., [16] for a brief account. It is clear that an estimate of the form (1) must hold for V.

We have

$$\begin{aligned} V(g(x)) &= l^T h\Big(g(x)\Big) \\ &= l^T h\Big(h^{-1}(Ph(x))\Big) \\ &= l^T P h(x)) \\ &< l^T h(x) = V(x). \end{aligned}$$

From here it follows as in the proof of Theorem III.3 that there exists a continuous positive definite function $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $V(g(x)) - V(x) \leq -\alpha(\|x\|)$ for all $x \in \mathbb{R}^n_+$. ■

Another result for the continuous-time domain is the following. First we need some new notation. Let $\mathcal{S}_{n,k}$ denote the set of all possible maps from $\{1, \ldots, n\}$ to $\{1, \ldots, k\}$ and denote by $A^{(j)}$ the jth column of a matrix A.

Theorem IV.8 *Let the matrices A_i, $i \in \{1, \ldots, k\}$ be all Hurwitz and Metzler. Assume that for every $x \in \mathbb{R}^n_+$ there exists an i such that*

$$f(x) \leq A_i x. \tag{22}$$

Assume further that the matrices

$$\left[A^{(1)}_{\sigma(1)} A^{(2)}_{\sigma(2)} \dots A^{(n)}_{\sigma(n)}\right] \tag{23}$$

are Hurwitz for every $\sigma \in \mathcal{S}_{n,k}$. Then there exists a vector $l \gg 0$ such that $W(x) = l^T x$ is a Lyapunov function with respect to (Σ_C).

Note that W can be written as

$$W(x) = \sum_i \int_0^{x_i} l_i \mathrm{d}s,$$

i.e., W is of the form (4).

Proof: The existence of $l \gg 0$ is in fact equivalent to condition (23), see [13, Thm.4]. Clearly there exist ψ_1, ψ_2 such that (1) holds. Now for every $x \geq 0$ there exists an index i such $f(x) \leq A_i x$, hence $D^+ W(x) = l^T f(x) \leq l^T A_i x < 0$. As in the proof of Theorem III.3 we can construct a continuous positive definite function $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that we have $D^+ W(x) \leq -\alpha(\|x\|)$ for all $x \geq 0$. ■

Remark IV.9 *In [9] for the case $n = 2$ the existence of an Ω-path used to construct scaling functions λ_i for the sum-type Lyapunov function.*

V. Applications to stability analysis of large-scale interconnections

The aim of this section is to briefly motivate, why smooth copositive Lyapunov functions given by a relatively simple formula are of particular interest.

In [18] for a more general setting than space permits here it has been shown that individual Lyapunov functions of subsystems in a possibly large-scale interconnection together with a smooth Lyapunov function for a comparison system yield a smooth Lyapunov function for the composite large-scale system. Of special interest is to find the Lyapunov function of the comparison system.

To this end consider n interconnected control systems of the form

$$\dot{X}_i = F_i(X_1, \dots, X_n), \quad X_i \in \mathbb{R}^{N_i}, \quad i = 1, \dots, n.$$

Assume certain smooth Lyapunov functions $\mathcal{V}_i\colon \mathbb{R}^{N_i} \to \mathbb{R}_+$ are given, such that with $\mathcal{V}(X) = (\mathcal{V}_1(X_1), \dots, \mathcal{V}_n(X_n))^T$ and $F(x)$ defined in the obvious way, it holds that $\langle \nabla \mathcal{V}, F(X) \rangle \leq f(\mathcal{V}(x))$ with $f\colon \mathbb{R}^n_+ \to \mathbb{R}^n$ locally Lipschitz and $f(0) = 0$. Then knowledge of a Lyapunov function $L\colon \mathbb{R}^n_+ \to \mathbb{R}_+$ for the comparison system

$$\dot{v} = f(v)$$

yields a Lyapunov function for the large-scale interconnection $\dot{X} = F(X)$ via $V(X) := L(\mathcal{V}(X))$. In particular, if L can be constructed in a smooth manner and the functions $\mathcal{V}_i$ are smooth then also V is smooth.

VI. Conclusions

In a unified way we have presented Lyapunov-type results for global asymptotic stability of the origin with respect to a monotone system which evolves in either continuous or discrete time. Several new Lyapunov function constructions and the so-called sum-type and max-type constructions from the literature have been presented for monotone systems. It was also shown how such Lyapunov functions for monotone systems can be utilized for the construction of a Lyapunov function of more general large-scale systems.

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